JOURNAL OF COMPUTATIONAL PHYSICS 116, 189–191 (1995)

NOTE

A Weak Formulation of Roe's Approximate Riemann Solver Applied to the St. Venant Equations

1. INTRODUCTION

Recently Toumi [1] presented a weak formulation of Roe's approximate Riemann solver based on a definition of a nonconservative product. Toumi first identifies the Lipschitz continuous path connecting two states that leads to the Roe-averaged state [2] for an ideal gas and then constructs a generalised Roe-averaged matrix for the Euler equations for real gases by using the same path. The purpose of this paper is to show that employing the ideas presented in [1] to the shallow water equations leads to the approximate Riemann solver given in [3].

2. SHALLOW WATER FLOWS

The one-dimensional shallow water equations can be written as

$$\mathbf{u}_t + \mathbf{f}_x = \mathbf{0}, \tag{2.1}$$

where

$$\mathbf{u} = (\rho, \rho u)^T \tag{2.2}$$

and

$$\mathbf{f} = (\rho u, \tfrac{1}{2}\rho^2 + \rho u^2)^T. \tag{2.3}$$

The quantities $(\rho, u) = (\rho, u)(x, t)$ represent the non-dimensional height and velocity at a general position x in space and at time t. Equations (2.1)–(2.3) are sometimes referred to as the St. Venant equations.

3. AN APPROXIMATE RIEMANN SOLVER (WEAK FORMULATION)

In [1] it is proposed that we solve equations of the form (2.1) via locally linearised Riemann problems

$$\mathbf{u}_t + A(\mathbf{u}_L, \mathbf{u}_R)_\Phi \mathbf{u}_x = \mathbf{0} \tag{3.1}$$

$$\mathbf{u}(x, 0) = \begin{cases} \mathbf{u}_L & \text{if } x < 0 \\ \mathbf{u}_R & \text{if } x > 0, \end{cases} \tag{3.2}$$

where $A(\mathbf{u}_L, \mathbf{u}_R)_\Phi$ *is a constant matrix which depends on the* data $(\mathbf{u}_L, \mathbf{u}_R)$ and on the path $\Phi(s; \mathbf{u}_L, \mathbf{u}_R)$, and which satisfies

$$\int_0^1 A(\Phi(s; \mathbf{u}_L, \mathbf{u}_R)) \frac{\partial \Phi}{\partial s}(s; \mathbf{u}_L, \mathbf{u}_R)\, ds = A(\mathbf{u}_L, \mathbf{u}_R)_\Phi (\mathbf{u}_R - \mathbf{u}_L) \tag{3.3}$$

$$A(\mathbf{u}, \mathbf{u})_\Phi = A(\mathbf{u}) \tag{3.4}$$

and

$$A(\mathbf{u}_L, \mathbf{u}_R)_\Phi \text{ has real eigenvalues and a complete set of eigenvectors'} \tag{3.5}$$

where

$$A = \partial \mathbf{f} / \partial \mathbf{u}$$

is the Jacobian of $\mathbf{f}$. (N.B. This also applies to nonconservative systems of the form $\mathbf{u}_t + A(\mathbf{u})\mathbf{u}_x = \mathbf{0}$. However, when the system is conservative, as is the case here, (3.3) is equivalent to the condition $\mathbf{f}(\mathbf{u}_R) - \mathbf{f}(\mathbf{u}_R) = A(\mathbf{u}_L, \mathbf{u}_R)_\Phi (\mathbf{u}_R - \mathbf{u}_L)$, which was originally proposed by Roe [2].)

As noted by Roe [2], the canonical path (a straight line) linking $\mathbf{u}_L$ and $\mathbf{u}_R$,

$$\Phi(s; \mathbf{u}_L, \mathbf{u}_R), = \mathbf{u}_L + s(\mathbf{u}_R - \mathbf{u}_L), \quad s \,\varepsilon\, [0, 1], \tag{3.6}$$

gives

$$A(\mathbf{u}_L, \mathbf{u}_R)_\Phi = \int_0^1 A(\mathbf{u}_L + s(\mathbf{u}_R - \mathbf{u}_L))\, ds, \tag{3.7}$$

which will, in general, involve integrals which may not emerge in closed form, or the closed form may be expensive to compute. The alternative approach adopted by Roe is to introduce a parameter vector $\mathbf{w}$, and it is shown in [1] that the choice of the canonical path for $\mathbf{w}$ leads to Roe's original scheme for the Euler equations with ideal gases [2]. This choice is then employed in the case of real gases to lead to a new scheme [1].

The Riemann solver in [1] is constructed by letting $\mathbf{f}_0$ be a smooth function such that $\mathbf{f}_0(\mathbf{w}_L) = \mathbf{u}_L$, $\mathbf{f}_0(\mathbf{w}_R) = \mathbf{u}_R$, and

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$A_0(\mathbf{w}) = \partial \mathbf{f}_0/\partial \mathbf{w}$ is a regular matrix for every state $\mathbf{w}$. The path chosen linking the two states $\mathbf{u}_L$ and $\mathbf{u}_R$ is then

$$\mathbf{\Phi}_0(s; \mathbf{u}_L, \mathbf{u}_R) = \mathbf{f}_0(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L)), \tag{3.8}$$

and this leads to the Roe matrix

$$A(\mathbf{u}_L, \mathbf{u}_R)_{\Phi_0} = C(\mathbf{u}_L, \mathbf{u}_R)_{\Phi_0} B(\mathbf{u}_L, \mathbf{u}_R)_{\Phi_0}^{-1}, \tag{3.9}$$

where

$$B(\mathbf{u}_L, \mathbf{u}_R)_{\Phi_0} = \int_0^1 A_0 (\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L))\, ds \tag{3.10}$$

and

$$C(\mathbf{u}_L, \mathbf{u}_R)_{\Phi_0} = \int_0^1 A(\mathbf{f}_0(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L)) A_0(\mathbf{w}_L + s(\mathbf{w}_R - {}_L))\, ds \tag{3.11}$$

which satisfies (3.3)–(3.5).

Our aim now is to show that the application of this Riemann solver to the equations of flow in Section 2 leads to the Riemann solver given in [3].

4. APPLICATION TO SHALLOW WATER FLOWS

For Eqs. (2.1)–(2.3), with parameter vector

$$\mathbf{w} = (w_1, w_2)^{\mathrm{T}} = (\sqrt{\rho}, \sqrt{\rho}\, u)^{\mathrm{T}}, \tag{4.1}$$

then

$$\mathbf{f}_0(\mathbf{w}) = \mathbf{u} = (\rho, \rho u)^{\mathrm{T}} = (w_1^2, w_1 w_2)^{\mathrm{T}}, \tag{4.2}$$

so that

$$A_0 = \frac{\partial \mathbf{f}_0}{\partial \mathbf{w}} = \begin{bmatrix} 2w_1 & 0 \\ w_2 & w_1 \end{bmatrix}. \tag{4.3}$$

From (3.10) and (4.3)

$$B(\mathbf{u}_L, \mathbf{u}_R)_{\Phi_0} = \int_0^1 A_0(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L))\, ds = \begin{bmatrix} 2\overline{w}_1 & 0 \\ \overline{w}_2 & \overline{w}_1 \end{bmatrix}, \tag{4.4}$$

where the overbar denotes the arithmetic mean of left and right states, $\overline{\mathbf{w}} = \frac{1}{2}(\mathbf{w}_L + \mathbf{w}_R)$. To construct the matrix $C(\mathbf{u}_L, \mathbf{u}_R)_{\Phi_0}$ (having found $B(\mathbf{u}_L, \mathbf{u}_R)_{\Phi_0}$), and hence $A(\mathbf{u}_L, \mathbf{u}_R)_{\Phi_0}$, it is necessary to write the Jacobian

$$A = \frac{\partial \mathbf{f}}{\partial \mathbf{u}} = \begin{bmatrix} 0 & 1 \\ \rho - u^2 & 2u \end{bmatrix} \tag{4.5}$$

as a function of $\mathbf{w}$:

$$A(\mathbf{u}(\mathbf{w})) = \begin{bmatrix} 0 & 1 \\ w_1^2 - \dfrac{w_2^2}{w_1^2} & \dfrac{2w_2}{w_1} \end{bmatrix}. \tag{4.6}$$

Combining (4.3) and (4.6) gives

$$A(\mathbf{u}(\mathbf{w}))\, A_0(\mathbf{w}) = \begin{bmatrix} w_2 & w_1 \\ 2w_1^3 & 2w_2 \end{bmatrix}, \tag{4.7}$$

so that from (3.11),

$$C(\mathbf{u}_L, \mathbf{u}_R)_{\Phi_0} = \int_0^1 A(\mathbf{f}_0(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L)))\, A_0(\mathbf{w}_L + s(\mathbf{w}_R - \mathbf{w}_L))\, ds = \begin{bmatrix} \overline{w}_2 & \overline{w}_1 \\ 2\widetilde{w_1^3} & 2\overline{w}_2 \end{bmatrix}, \tag{4.8}$$

where again $\overline{w} = \frac{1}{2}(\mathbf{w}_L + \mathbf{w}_R)$ denotes the arithmetic mean and $\widetilde{w_1^3}$ is an approximation to w_1^3 given by

$$\widetilde{w_1^3} = \int_0^1 (w_{1_L} + s(w_{1_R} - w_{1_L}))^3\, ds$$

$$= \begin{cases} \dfrac{w_{1_R}^4 - w_{1_L}^4}{4(w_{1_R} - w_{1_L})} & \text{if } w_{1_R} \neq w_{1_L} \quad (4.9a) \\ w_{1_L}^3\, (= w_{1_R}^3) & \text{if } w_{1_R} = w_{1_L} \quad (4.9b) \end{cases}$$

However, since

$$w_{1_R}^4 - w_{1_L}^4 = (w_{1_R} - w_{1_L})(w_{1_R}^3 + w_{1_R}^2 w_{1_L} + w_{1_R} w_{1_L}^2 + w_{1_L}^3), \tag{4.10}$$

then (4.9a) and (4.9b) become

$$\widetilde{w_1^3} = \tfrac{1}{4}(w_{1_R}^3 + w_{1_R}^2 w_{1_L} + w_R^1 w_{1_L}^2 + w_{1_L}^3). \tag{4.11}$$

Combining (4.4), (4.8), and (4.11), we find that the matrix in (3.9) for the system of equations under consideration here is

$$A(\mathbf{u}_L, \mathbf{u}_R)_{\Phi_0} = C(\mathbf{u}_L, \mathbf{u}_R)_{\Phi_0} B(\mathbf{u}_L, \mathbf{u}_R)_{\Phi_0}^{-1}$$

$$= \begin{pmatrix} 0 & 1 \\ \frac{\widetilde{w_1^3}}{\overline{w}_1} - \frac{\overline{w}_2^2}{\overline{w}_1^2} & \frac{2\overline{w}_2}{\overline{w}_1} \end{pmatrix}.$$

Thus, since

$$\frac{\overline{w}_2}{\overline{w}_1} = \frac{\sqrt{\rho_R}\, u_R + \sqrt{\rho_L}\, u_L}{\sqrt{\rho_R} + \sqrt{\rho_L}} = \tilde{u},$$

say, and

$$\begin{aligned} \widetilde{w_1^3} &= \tfrac{1}{4}(w_{1_R}^3 + w_{1_R}^2 w_{1_L} + w_{1_R} w_{1_L}^2 + w_{1_L}^3) = \tfrac{1}{4}(w_{1_R} + w_{1_L})(w_{1_R}^2 + w_{1_L}^2) \\ &= \overline{w}_1 \overline{w_1^2} \end{aligned}$$

then

$$A(\mathbf{u}_L, \mathbf{u}_R)_{\Phi_0} = \begin{pmatrix} 0 & 1 \\ \overline{\rho} - \tilde{u}^2 & 2\tilde{u} \end{pmatrix},$$

where

$$\overline{\rho} = \tfrac{1}{2}(\rho_L + \rho_R) = \tfrac{1}{2}(w_{1_L}^2 + w_{1_R}^2),$$

again denotes the arithmetic mean, which is precisely the Roe matrix given in [3], and clearly represents an approximation to the Jacobian (4.5).

CONCLUSIONS

We have demonstrated that the Riemann solver proposed by Toumi [1], when applied to the St. Venant equations representing shallow water flows, results in the Riemann solver given by Glaister [3].

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Received December 5, 1990; revised August 24, 1994

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